

Matrix model approach to minimal Liouville gravity revisited

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ABSTRACT. Using the connection with the Frobenius manifold structure, we study the matrix model description of minimal Liouville gravity (MLG) based on the Douglas string equation. Our goal is to find an exact discrete formulation of the (q, p) MLG model that intrinsically contains information about the conformal selection rules. We discuss how to modify the Frobenius manifold structure appropriately for this purposes. We propose a modification of the construction for Lee–Yang series involving the A_{p-1} algebra instead of the previously used A_1 algebra. With the new prescription, we calculate correlators on the sphere up to four points and find full agreement with the continuous approach without using resonance transformations.

1. INTRODUCTION

Minimal Liouville gravity (MLG) is a special model of Liouville gravity [1] with the matter sector represented by the (q, p) minimal CFT model.¹ MLG is a BRST theory with the well-known structure of the Hilbert space [2]. One of the main problems in the theory is to compute the correlators of the primary cohomologies. Because of the integrals over moduli involved in the construction of the correlators, this computation requires quite sophisticated techniques [3]. Only expressions up to four-point correlation numbers have been found explicitly so far [2, 4]. Here, we study an alternative approach to MLG that gives a simple procedure for computing the correlation functions. This approach is connected with the matrix model (MM) [5–12] of two-dimensional quantum gravity and also called a discrete approach. The basic

¹In this paper, we focus on the A -series of Virasoro minimal models.

fact about the MLG–MM correspondence is the coincidence of the spectra of the gravitational dimensions [13]. This result represents the main support of the idea that the two approaches describe the same quantum theory. In the (q, p) model² of MLG, the primary BRST cohomologies O_{mn} are labeled by two integers $m = 1, \dots, q - 1$ and $n = 1, \dots, p - 1$. In what follows, we assume $q < p$. Hence, the main object of our study is the generating function of the correlators of the primary cohomologies

$$Z = \langle \exp \sum_n \lambda_{mn} O_{mn} \rangle_{\text{MLG}} = \sum_{N=0}^{\infty} \sum_{n_i} \frac{\lambda_{m_1 n_1} \cdots \lambda_{m_N n_N}}{N!} \langle O_{m_1 n_1} \cdots O_{m_N n_N} \rangle_{\text{MLG}}. \quad (1.1)$$

The brackets $\langle \dots \rangle_{\text{MLG}}$ denote the integrated correlation functions, which we therefore call correlation numbers. In what follows, we call the parameters λ_{mn} Liouville coupling constants.

As was first shown in the seminal KPZ paper [13], the scaling properties of the correlators in MLG are governed by the following rule. The dependence of the correlator $G = \langle O_{m_1 n_1} \cdots O_{m_N n_N} \rangle$ on the cosmological constant is given by

$$G(\mu) = \mu^{\frac{p+q}{q} - \sum_{i=1}^N \delta_{m_i n_i}} G(1), \quad (1.2)$$

where

$$\delta_{mn} = \frac{p + q - |pm - qn|}{2q}, \quad O_{mn} \sim \mu^{-\delta_{mn}}. \quad (1.3)$$

Therefore, the first basic requirement for the dual approach is to reproduce this spectrum. In the dual description of the (q, p) MLG model, there are two basic elements. For the spherical topology, we introduce the polynomial

$$Q(y) = y^q + u_1 y^{q-2} + \cdots + u_{q-1}, \quad (1.4)$$

where y is an auxiliary variable (we discuss in more detail below, that this polynomial defines the structure of a special Frobenius manifold (FM) and the set $\{u_\alpha\}$ represents a special choice of coordinates on this manifold) [14, 15] and the so-called action

$$S(t_{mn}) = \text{res}_{y=\infty} \left(Q^{\frac{p+q}{q}} + \sum_{m,n}^{pm-qn>0} t_{mn} Q^{\frac{pm-qn}{q}} \right), \quad (1.5)$$

which defines the generating function of the correlation numbers and appears to be the subject of the string equation [16]. The parameters t_{mn} are known as KdV times (or couplings). Weights $\tilde{\delta}_{mn}$ ($\tilde{\delta}_{11} = 1$) can be assigned to the couplings t_{mn} and y so that $Q(y)$ becomes quasihomogeneous of weight $1/2$. With the identification $t_{11} \sim \mu$, it can be easily verified [16] that the spectrum of gravitational dimensions is exactly the spectrum (1.3) that appears in the continuous approach to MLG, i.e., $\delta_{mn} = \tilde{\delta}_{mn}$. We thus obtain a natural identification between the couplings of the two approaches $t_{mn} = \lambda_{mn}$.

²We recall that q and p are two coprime integers.

After Douglas had shown [16] that the scaling dimensions of KdV times coincide with those of MLG coupling constants, there were attempts to verify the correspondence between the two approaches at the level of the correlation functions. However, the obtained correlators failed to satisfy conformal selection rules already at one-point level. A possible resolution of this problem was formulated by Moore, Seiberg, and Staudacher [19]. The idea was based on the observation that MLG by definition contains an ambiguity related to the fact that the correlation numbers, which are given by integrals over moduli spaces, depend on the contact behavior when the positions of a few insertions collide with each other. Indeed, such ultraviolet information is not provided in the standard CFT formalism based on the notion of the operator product expansion. This ambiguity allows extending the possible form of the relations between Liouville coupling constants and KdV times,

$$t_{mn} = \lambda_{mn} + \sum_{m_1 n_1 m_2 n_2} A_{mn}^{m_1 n_1, m_2 n_2} \lambda_{m_1 n_1} \lambda_{m_2 n_2} + \dots, \quad (1.6)$$

where the nonlinear terms are admissible only if they satisfy the resonance conditions

$$\delta_{mn} = \delta_{m_1 n_1} + \dots + \delta_{m_k n_k}. \quad (1.7)$$

This is why formula (1.6) is sometimes called the resonance transformation. The idea was to tune the parameters $A_{mn}^{m_1 n_1, \dots}$ of this transformation in order to satisfy the basic requirements of the MLG theory, namely, the conformal selection rules for the correlation numbers inherited from the minimal model [17] representing its matter sector.³

Nevertheless, it appears that except for the Lee–Yang series $(2, 2s+1)$ [18, 19], this program can be followed literally only up to two-point correlators. Although there is full agreement when the fusion rules are satisfied, a discrepancy appears in the *non-physical region*, i.e., when they are not satisfied. For example, there are three-point correlators, which must be zero according to the selection rules, but they cannot be made so using the resonance transformation. It was therefore conjectured that the dual description does not exactly correspond to the MLG models but describes somewhat modified theories.

Further progress in understanding the dual models was achieved after the relation of the Douglas approach to the (q, p) MLG with the A_{q-1} Frobenius manifold structure was revealed (see [20–23] and the references therein). In particular, it became clear that the generating function of the correlation numbers is just the tau function of the Gelfand–Dikij integrable hierarchy connected with the A_{q-1} Frobenius manifold.

³We note that because of the structure of the spectrum of the gravitational dimensions in each particular model, the form of the resonance transformations is highly constrained and the problem becomes quite nontrivial (see, e.g., [18]).

From this relation, we can derive the nice representation

$$Z = \frac{1}{2} \int_0^{v_*} C_{\alpha\beta\gamma}(v) \frac{\partial S}{\partial v_\beta} \frac{\partial S}{\partial v_\gamma} dv^\alpha, \quad (1.8)$$

which will be important for our purposes. Here, v_α ($\alpha = 1, \dots, q-1$) are the flat coordinates on the FM, and $C_{\alpha\beta\gamma}$ are the structure constants of the A_{q-1} Frobenius algebra, the algebra of polynomials modulo the ideal generated by the polynomial $\frac{dQ}{dy}$. We discuss the properties of the action S in the flat coordinates later. Perhaps the most important ingredient in (1.8) is the upper limit v_* , which is a special solution of the string equation

$$\frac{\partial S}{\partial v_\alpha}(v_*) = 0. \quad (1.9)$$

It was argued in [21] that only one solution of the string equation with the special property $v_{*\alpha}(\lambda_{mn}) = 0$ for $\alpha > 1$ and $\lambda_{mn} = 0$ (except $\lambda_{11} = \mu$) allows satisfying the conformal selection rules. After the transition to the flat coordinates, the necessary expressions for the structure constants were obtained in [22]. All these results made calculations in the discrete approach very clear both technically and conceptually.

As already mentioned, such formulated theories cannot be regarded as exactly the same MLG theories, because it is impossible to satisfy the selection rules for all correlation numbers. The natural question is what can be modified in order to obtain an exact discrete analogue of the MLG theory. The first thing that comes to mind is to analyze possible modifications of the relevant FM structures. Without going into detail, we note that a FM is a quite rigid construction that, in particular, intrinsically contains a special Milnor ring (see, e.g., [24]). It is quite natural to try a possible modification in the simplest case of the Lee–Yang series $(2, p)$. But in this case, we have $Q' = 2y$, and the corresponding algebra is trivial, $A_1 = 1$. According to the previous considerations, all physics in this case is concentrated in the form of resonance transformation (1.6). In fact, this seems rather strange because the relation between Milnor rings and Verlinde algebras (fusion rings) appearing in the conformal field theories makes us think that the information about selection rules should be encoded in the structure of the Frobenius algebra itself. For the $(2, p)$ series, for example, the dimensionality of the possible candidate should correlate with the dimensionality of the Kac table (or simply with p). A question arises here: Can we use the A_{p-1} algebra to describe the $(2, p)$ series of MLG models? The first answer is no, because we must construct at least the same spectrum of gravitational dimensions as we had using A_1 . But we note that the spectrum depends not only on the quasihomogeneity property of the polynomial $Q(y)$ but also on the structure of the action S . In this paper we show that the modification can be made properly such that the spectrum of the scaling dimensions reproduces the spectrum that appears in the continuous formulation. Further, we calculate the correlation numbers up to

four-point correlators and show that the results agree perfectly with the results of the continuous approach without any need for the resonance transformations.

2. CALCULATION OF ONE- AND TWO-POINT CORRELATION NUMBERS

We consider the series of $(2, p)$, ($p \geq 5$ and p is odd) minimal models coupled to Liouville gravity in the spherical topology. The primary fields are enumerated as $O_n = O_{p-n}$, $n \in [1, \frac{p-1}{2}]$. As discussed, we work with the polynomial $Q(y) = y^p + u_1 y^{p-2} + \dots + u_{p-1}$ instead of $Q(y) = y^2 + u_1$. We first ensure that we obtain spectrum (1.3), i.e., $O_k \sim \mu^{-\frac{k+1}{2}}$. The action S with the appropriate scaling properties is

$$S = \text{res}_{y=\infty} \left(Q^{1+\frac{2}{p}} + \sum_{n=1}^{2n < p} \lambda_n Q^{\frac{p-2n}{p}} \right). \quad (2.1)$$

Because $Q^{1+\frac{2}{p}} \sim \mu Q^{\frac{p-2}{p}}$ and $Q^{1+\frac{2}{p}} \sim \lambda_k Q^{\frac{p-2k}{p}}$, we have $\lambda_k \sim O_k^{-1} \sim Q^{\frac{2+2k}{p}} \sim \mu^{\frac{k+1}{2}}$. Using the definition

$$\theta_{\alpha,k} = \text{res}_{y=\infty} Q^{k+\frac{\alpha}{p}}(y), \quad (2.2)$$

we can rewrite our action in terms of $\theta_{\alpha,k}$:

$$S = \theta_{2,1} + \mu \theta_{p-2,0} + \sum_{n=2}^{\frac{p-1}{2}} \lambda_n \theta_{p-2n,0}, \quad (2.3)$$

where $\mu = \lambda_1$. In what follows, we use the proposition [21]

$$\begin{cases} k \text{ even:} & \frac{\partial \theta_{\lambda,k}}{\partial v_\alpha} = \delta_{\lambda,\alpha} x_{\lambda,k} \left(-\frac{v_1}{p} \right)^{\frac{k}{2}p}, \\ k \text{ odd:} & \frac{\partial \theta_{\lambda,k}}{\partial v_\alpha} = \delta_{\lambda,p-\alpha} y_{\lambda,k} \left(-\frac{v_1}{p} \right)^{\frac{k-1}{2}p+\lambda}, \end{cases} \quad (2.4)$$

where

$$x_{\lambda,k} = \frac{\Gamma\left(\frac{\lambda}{p}\right)}{\Gamma\left(\frac{\lambda}{p} + \frac{k}{2}\right) \left(\frac{k}{2}\right)!} \quad \text{and} \quad y_{\lambda,k} = -\frac{\Gamma\left(\frac{\lambda}{p}\right)}{\Gamma\left(\frac{\lambda}{p} + \frac{k+1}{2}\right) \left(\frac{k-1}{2}\right)!}. \quad (2.5)$$

In the same way as in [23], we can obtain the formulas for the structure constant

$$C_{\alpha\beta\gamma} = \left(-\frac{v_1}{p} \right)^{\frac{\alpha+\beta+\gamma-p-1}{2}} \theta(\alpha, \beta, \gamma), \quad (2.6)$$

where $\theta(\alpha, \beta, \gamma) = 1 \Leftrightarrow \alpha \in [|\beta + \gamma - p| + 1 : 2 : p - 1 - |\beta - \gamma|]$, and its derivative in the flat coordinates on the line $v_{\alpha>0} = 0$,

$$\partial_\delta C_{\alpha\beta\gamma} = \theta(\alpha, \beta, \gamma, \delta, p) \frac{2p - \alpha - \beta - \gamma - \rho}{2p} \left(-\frac{v_1}{p} \right)^{\frac{\alpha+\beta+\gamma+\rho-2p-2}{2}}, \quad (2.7)$$

if $(\alpha + \beta + \gamma + \rho - 2p - 2)/2 \in \mathbb{N}_0$, where

$$\begin{aligned} \theta(\alpha, \beta, \gamma, \delta, p) = & [(p - m_1)\chi_{1,m_1}(m_2 + m_3 - m_4) + \frac{2p + m_4 - m_1 - m_2 - m_3}{2} \times \\ & \times \chi_{m_1+2,2p-m_1-2}(m_2 + m_3 - m_4)], \end{aligned} \quad (2.8)$$

and $m_i = \text{RankedMax}[\{\alpha, \beta, \gamma, \delta\}, i]$.

We now calculate one-point numbers. As we know from the conformal selection rules, the one-point correlation numbers of all operators except the unity operator must be zero. Indeed, we find

$$Z_1 = \langle O_n \rangle = \int_0^{v_{*1}} C_{p-1,\alpha,\beta} \frac{\partial S^{(0)}}{\partial v_\alpha} \frac{\partial S^{(n)}}{\partial v_\beta} dv_1. \quad (2.9)$$

Taking into account that

$$C_{p-1,\alpha,\beta} \sim \delta_{\alpha,\beta}, \quad \frac{\partial S^{(0)}}{\partial v_\alpha} \sim \delta_{\alpha,p-2}, \quad \frac{\partial S^{(n)}}{\partial v_\alpha} \sim \delta_{\alpha,p-2n}, \quad (2.10)$$

it follows that the one-point function $Z_1 = 0$ for $n \neq 1$. In particular, in the case $n = 1$, we find

$$Z_0 = \langle O_1 \rangle = \frac{1}{2} \int_0^{v_{*1}} C_{p-1,\alpha,\beta} \frac{\partial S^{(0)}}{\partial v_\alpha} \frac{\partial S^{(0)}}{\partial v_\beta} dv_1 = \frac{1}{2} \int_0^{v_{*1}} C_{p-1,p-2,p-2} \left(y_{2,1} \left(\frac{v_1}{p} \right)^2 + \mu \right)^2 dv_1. \quad (2.11)$$

Substituting $\mu = \frac{v_{*1}^2}{2p}$, we can obtain

$$Z_0 = \frac{1}{(p-2)p(p+2)} \frac{v_{*1}^{p+2}}{p^{p+3}}. \quad (2.12)$$

And the two-point correlator is

$$Z_{12} = \langle O_{n_1} O_{n_2} \rangle = \int_0^{v_{*1}} C_{p-1,\alpha,\beta} \frac{\partial S^{(n_1)}}{\partial v_\alpha} \frac{\partial S^{(n_2)}}{\partial v_\beta} dv_1 = \delta_{n_1,n_2} \left(-\frac{1}{p} \right)^{p-2n_1-1} \frac{v_{*1}^{p-2n_1}}{p-2n_1}. \quad (2.13)$$

Hence, the two-point correlators have a proper diagonal form. On the other hand, a simple analysis based on the new action S shows that there is no way to use the freedom of the resonance transformations in this case.

3. CALCULATION OF THREE-POINT CORRELATORS

In this section, we calculate three-point numbers and compare the resulting normalized expression with the expression from the continuous approach,

$$Z_{123} = \langle O_{n_1} O_{n_2} O_{n_3} \rangle = C_\gamma^{\alpha\beta} \frac{\partial v_*^\gamma}{\partial \lambda_{n_3}} \frac{\partial S^{(n_1)}}{\partial v_\alpha} \frac{\partial S^{(n_2)}}{\partial v_\beta} = C_{\gamma,p-2n_1,p-2n_2} \frac{\partial v_*^\gamma}{\partial \lambda_{n_3}}. \quad (3.1)$$

To obtain an expression for $\frac{\partial v_*^\gamma}{\partial \lambda_{n_3}}$, we proceed as follows. We use the Douglas string equation as

$$\left. \frac{\partial}{\partial \lambda_k} \frac{\partial S}{\partial v^\alpha} \right|_{v_*} = 0, \quad \alpha = 1, \dots, p-1. \quad (3.2)$$

Substituting (2.3), we have

$$\frac{\partial}{\partial \lambda_k} \left[\frac{\partial \theta_{2,1}}{\partial v^\alpha} + \mu \frac{\partial \theta_{p-2,0}}{\partial v^\alpha} + \sum_{n=2}^{\frac{p-1}{2}} \lambda_n \frac{\partial \theta_{p-2n,0}}{\partial v^\alpha} \right] = 0. \quad (3.3)$$

Hence, we obtain the following useful equation

$$C_{2\alpha\beta} \frac{\partial v_*^\beta}{\partial \lambda_k} + \delta_{\alpha,2k} = 0. \quad (3.4)$$

Taking the special nonzero condition for $C_{2\alpha\beta}$ into account, namely, $C_{2\alpha\beta} \neq 0$ iff $\alpha + \beta = p \pm 1$, we can recursively obtain an expression for $\frac{\partial v_*^\beta}{\partial \lambda_k}$. Indeed, considering (3.4) for $\alpha = 1$ and for $\alpha = p-1$, we obtain

$$\frac{\partial v_*^{p-2}}{\partial \lambda_k} = 0, \quad \frac{\partial v_*^2}{\partial \lambda_k} = \frac{p}{v_{*1}} \delta_{k, \frac{p-1}{2}}. \quad (3.5)$$

If $\alpha \neq p-1$ and $\alpha \neq 1$, then $\beta = p+1-\alpha$ or $\beta = p-1-\alpha$, and we obtain

$$\frac{\partial v_*^{p+1-\alpha}}{\partial \lambda_k} = \frac{p}{v_{*1}} \left(\frac{\partial v_*^{p-1-\alpha}}{\partial \lambda_k} + \delta_{\alpha,2k} \right) \quad (3.6)$$

from (3.4). By recursive computation, we can get zero for odd β and for even β :

$$\frac{\partial v_*^\beta}{\partial \lambda_k} = \left(\frac{p}{v_{*1}} \right)^{\frac{\beta+2k+1-p}{2}} \quad \text{if} \quad \frac{\beta}{2} \geq \frac{p+1}{2} - k. \quad (3.7)$$

Combining (2.6) and (3.7), we obtain

$$Z_{123} = \left(\frac{v_{*1}}{p} \right)^{p-1-\sum_{i=1}^3 n_i} \sum_{n=\max\{1, \frac{p+1}{2}-n_3\}}^{\frac{p-1}{2}} (-1)^{\frac{p-1}{2}-n_1-n_2+n} \theta(2n, p-2n_1, p-2n_2). \quad (3.8)$$

After some operations with this expression, we finally find

$$Z_{123} = \left(\frac{v_{*1}}{p} \right)^{p-1-\sum_{i=1}^3 n_i} \theta_{123}, \quad (3.9)$$

where θ_{123} denotes the nonzero condition of this expression. It turns out that it coincides with the selection rules for three-point correlators,⁴ which come from the CFT fusion rules, i.e., $\theta_{123} = 1$ if $n_3 \in [|n_1 - n_2| + 1 : 2 : n_1 + n_2 - 1]$ or $p - n_3 \in [|n_1 - n_2| + 1 : 2 : n_1 + n_2 - 1]$.

⁴In the next condition, there is no minimum in the upper limits as soon as we take $n_i \in [1, \frac{p-1}{2}]$.

We now have all necessary ingredients for comparing with the continuous approach. Using (2.12), (2.13), and (3.9), we find the normalized expression for three-point numbers

$$\frac{Z_{123}^2 Z_0}{Z_{11} Z_{22} Z_{33}} = \frac{\prod_{i=1}^3 (p - 2n_i)}{(p-2)p(p+2)} \theta_{123}. \quad (3.10)$$

This is exactly the same as the continuous expression calculated in [4] for general (q, p) models

$$\frac{\langle \langle O_{m_1 n_2} O_{m_2 n_2} O_{m_3 n_3} \rangle \rangle^2}{\prod_{i=1}^3 \langle \langle O_{m_i n_i}^2 \rangle \rangle} = \frac{\prod_{i=1}^3 |m_i p - n_i q|}{p(p+q)(p-q)} \theta_{123}^{pq}, \quad (3.11)$$

where θ_{123}^{pq} denotes the selection rules of general (q, p) models.

4. CALCULATION OF FOUR-POINT CORRELATORS

In this section, we calculate four-point numbers and compare their normalized expression with the expressions found in the continuous approach:

$$\begin{aligned} Z_{1234}^{disc} &= \frac{\partial^2 v_*^\gamma}{\partial \lambda_3 \partial \lambda_4} C_\gamma^{\alpha\beta} \frac{\partial S^{(n_1)}}{\partial v^\alpha} \frac{\partial S^{(n_2)}}{\partial v^\beta} + \frac{\partial v_*^\gamma}{\partial \lambda_3} \frac{\partial C_\gamma^{\alpha\beta}}{\partial \lambda_4} \frac{\partial S^{(n_1)}}{\partial v^\alpha} \frac{\partial S^{(n_2)}}{\partial v^\beta} = \\ &= \frac{\partial^2 v_*^\gamma}{\partial \lambda_3 \partial \lambda_4} C_{\gamma, p-2n_1, p-2n_2} + \partial_\delta C_{\gamma, p-2n_1, p-2n_2} \frac{\partial v_*^\gamma}{\partial \lambda_3} \frac{\partial v_*^\delta}{\partial \lambda_4}. \end{aligned} \quad (4.1)$$

To obtain an expression for $\frac{\partial^2 v_*^\gamma}{\partial \lambda_3 \partial \lambda_4}$, we proceed the same way as for calculating $\frac{\partial v_*^\beta}{\partial \lambda_k}$. Differentiating (3.4) with respect to λ_j , we obtain

$$C_{2\alpha\beta} \frac{\partial^2 v_*^\beta}{\partial \lambda_j \partial \lambda_k} + \partial_\gamma C_{2\alpha\beta} \frac{\partial v_*^\gamma}{\partial \lambda_j} \frac{\partial v_*^\beta}{\partial \lambda_k} = 0. \quad (4.2)$$

Noting that $\partial_\gamma C_{2\alpha\beta} = -\frac{1}{p} \delta_{\alpha+\beta+\gamma, 2p}$, we rewrite (4.2) as

$$C_{2\alpha\beta} \frac{\partial^2 v_*^\beta}{\partial \lambda_j \partial \lambda_k} - \frac{1}{p} \left(\frac{p}{v_{*1}} \right)^{n_j + n_k - \frac{p-1}{2}} \sum_{n=\frac{p+1}{2}-n_j}^{\frac{p-1}{2}} \sum_{m=\frac{p+1}{2}-n_k}^{\frac{p-1}{2}} \left(\frac{p}{v_{*1}} \right)^{n+m} \delta_{\alpha, 2p-2n-2m}. \quad (4.3)$$

Using the ansatz

$$\frac{\partial^2 v_*^\beta}{\partial \lambda_j \partial \lambda_k} = -\frac{1}{p} \left(\frac{p}{v_{*1}} \right)^{n_j + n_k - \frac{p-1}{2}} \sum_{n=\frac{p+1}{2}-n_j}^{\frac{p-1}{2}} \sum_{m=\frac{p+1}{2}-n_k}^{\frac{p-1}{2}} f(\beta, p-n-m), \quad (4.4)$$

in the same way as we obtained result (3.7), we obtain

$$\frac{\partial^2 v_*^\beta}{\partial \lambda_j \partial \lambda_k} = -\frac{1}{p} \left(\frac{p}{v_{*1}} \right)^{n_j + n_k - \frac{p-3}{2} + \frac{\beta}{2}} \sum_{n=\frac{p+1}{2}-n_j}^{\frac{p-1}{2}} \sum_{m=\frac{p+1}{2}-n_k}^{\frac{p-1}{2}} \sum_{i=1}^{\beta/2} \delta_{n+m, \frac{p-1}{2}+i} \quad (4.5)$$

for even β . Combining (2.6) and (4.5), we can see that the first term in (4.1) is

$$\frac{\partial^2 v_*^\gamma}{\partial \lambda_3 \partial \lambda_4} C_{\gamma, p-2n_1, p-2n_2} = \frac{1}{p} \left(\frac{p}{v_{*1}} \right)^{\sum_{i=1}^4 n_i + 2 - p} \sum_{t=|\frac{p}{2}-n_1-n_2|+\frac{1}{2}}^{\frac{p-1}{2}-|n_1-n_2|} (-1)^{\frac{p+1}{2}-n_1-n_2+t} \sum_{i=1}^t \varphi(i), \quad (4.6)$$

where

$$\varphi(i) = \sum_{n=\frac{p+1}{2}-n_3}^{\frac{p-1}{2}} \sum_{m=\frac{p+1}{2}-n_4}^{\frac{p-1}{2}} \delta_{m+n, \frac{p-1}{2}+i} \quad (4.7)$$

or, explicitly,

$$\varphi(i) = \frac{p+1-2i}{4} + \frac{|n_3+n_4-\frac{p+1}{2}+i|}{2} - \frac{|n_3-\frac{p+1}{2}+i|}{2} - \frac{|n_4-\frac{p+1}{2}+i|}{2}. \quad (4.8)$$

Using (2.7), (3.7), and (4.6), we obtain the normalized expression

$$\frac{Z_{1234}^{disc} Z_0}{\sqrt{Z_{11} Z_{22} Z_{33} Z_{44}}} = \frac{\sqrt{\prod_i (p-2n_i)}}{(p-2)p(p+2)} \times \left[\sum_{t=|\frac{p}{2}-n_1-n_2|+\frac{1}{2}}^{\frac{p-1}{2}-|n_1-n_2|} (-1)^{\frac{p+1}{2}-n_1-n_2+t} \sum_{i=1}^t \varphi(i) + \sum_{n=\frac{p+1}{2}-n_3}^{\frac{p-1}{2}} \sum_{m=\frac{p+1}{2}-n_4}^{\frac{p-1}{2}} F(n_1, n_2, n, m, p) \right], \quad (4.9)$$

where

$$F(n_1, n_2, n, m, p) = (m+n-n_1-n_2)(-1)^{n+m-n_1-n_2} \theta(2n, 2m, p-2n_1, p-2n_2). \quad (4.10)$$

It is easy to verify that (4.9) is symmetric. In the continuous approach, for $n_1 \leq \dots \leq n_4$, this quantity is

$$\frac{Z_{1234}^{cont} Z_0}{\sqrt{Z_{11} Z_{22} Z_{33} Z_{44}}} = \frac{\sqrt{\prod_i (p-2n_i)}}{2(p-2)p(p+2)} \left(\sum_{i=2}^4 \sum_{t=-(n_1-1)}^{n_1-1} |p-2(n_i-t)| - n_1(p+2n_1) \right). \quad (4.11)$$

We recall that (4.11) is obtained [2] under the assumption that the number of conformal blocks of the four-point correlator is exactly n_1 . This can be expressed through the following condition

$$\begin{cases} n_1 + n_2 + n_3 + n_4 & \text{even:} & n_1 + n_4 \leq n_2 + n_3, \\ n_1 + n_2 + n_3 + n_4 & \text{odd:} & -n_1 + n_2 + n_3 + n_4 \geq p-2 \end{cases} \quad (4.12)$$

(which in turn ensures that the selection rules are satisfied). We find that in this region, (4.9) coincides with (4.11), while outside of this region our expression (4.9) gives zero values. It is interesting that the previous consideration based on the polynomial Q of the second degree in y gives sometimes nonzero values outside (4.12) [20], which makes these two results significantly different. Unfortunately, corresponding results

derived in the continuous approach are currently unknown, and we cannot conclude which of these two discrete versions is appropriate.

5. CONCLUSION

We have considered another description of the Lee–Yang series of MLG models. At the level of the correlation functions up to four points, we verified that the results obtained using this description agree with the results obtained using the initial continuous definition of MLG. The computation differs significantly from the previous one [18–20]. In particular, it does not require any resonance transformation. The essential modification is related to another choice of the FM relevant for the (q, p) MLG model. The construction for the $(2, p)$ model is now based on the A_{p-1} Frobenius algebra. We find this description more natural because the dimension of the A_{p-1} algebra exactly equals the number of basic physical BRST cohomologies constructed from the primary fields in the minimal model $(2, p)$ (modulo the standard symmetry factor 2 of the Kac table). It can therefore play the role of the regulator ensuring the necessary satisfaction of the selection rules, while in the previous scheme this role had to be solved by supplementary resonance transformations.

It would be interesting to seek a possible alternative description without resonances in the general (q, p) case. Certainly, this question is related to the longstanding problem of p – q duality in the discrete approach to MLG [25, 26]. Apparently, an appropriate description should be symmetric under the interchange of q and p .

Another interesting issue is related to the problem described at the end of the preceding section: the origin of the discrepancy between the two versions in the region outside (4.12). The question of computing (spherical and higher-genera) multipoint correlators (and also the correlators of the gravitational descendants) deserves to be studied in this perspective. This problem, in particular, requires more detailed analysis of the structure constants of the A_{p-1} Frobenius algebra in the flat coordinates. We plan to investigate this question in the near future.

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REFERENCES

- [1] A. M. Polyakov, *Quantum Geometry of Bosonic Strings*, *Phys.Lett.* **B103** (1981) 207–210.
- [2] A. Belavin and Al. Zamolodchikov, *Integrals over moduli spaces, ground ring, and four-point function in minimal Liouville gravity*, *Theor.Math.Phys.* **147** (2006) 729–754.
- [3] A. Zamolodchikov, *Higher equations of motion in Liouville field theory*, *Int. J. Mod. Phys.* **A 19S2** (2004) 510.

- [4] Al. Zamolodchikov, *Three-point function in the minimal Liouville gravity*, *Theor.Math.Phys.* **142** (2005) 183-196.
- [5] V. Kazakov, A. A. Migdal, and I. Kostov, *Critical Properties of Randomly Triangulated Planar Random Surfaces*, *Phys.Lett.* **B157** (1985) 295–300.
- [6] V. Kazakov, *Ising model on a dynamical planar random lattice: Exact solution*, *Phys.Lett.* **A119** (1986) 140–144.
- [7] V. Kazakov, *The Appearance of Matter Fields from Quantum Fluctuations of 2D Gravity*, *Mod.Phys.Lett.* **A4** (1989) 2125.
- [8] M. Staudacher, *The Yang-Lee edge singularity on a dynamical planar random surface*, *Nucl.Phys.* **B336** (1990) 349.
- [9] E. Brezin and V. Kazakov, *Exactly solvable field theories of closed strings*, *Phys.Lett.* **B236** (1990) 144–150.
- [10] M. R. Douglas and S. H. Shenker, *Strings in Less Than One-Dimension*, *Nucl.Phys.* **B335** (1990) 635.
- [11] D. J. Gross and A. A. Migdal, *Nonperturbative Two-Dimensional Quantum Gravity*, *Phys.Rev.Lett.* **64** (1990) 127.
- [12] I. Krichever, *The Dispersionless Lax equations and topological minimal models*, *Commun.Math.Phys.* **143** (1992) 415–429.
- [13] V. Knizhnik, A. M. Polyakov, and A. Zamolodchikov, *Fractal Structure of 2D Quantum Gravity*, *Mod.Phys.Lett.* **A3** (1988) 819.
- [14] B. Dubrovin, *Integrable systems in topological field theory*, *Nucl.Phys.* **B379** (1992) 627–689.
- [15] R. Dijkgraaf, H. L. Verlinde, and E. P. Verlinde, *Topological strings in d less than 1*, *Nucl.Phys.* **B352** (1991) 59–86.
- [16] M. R. Douglas, *Strings in less than one-dimension and the generalized KdV hierarchies*, *Phys.Lett.* **B238** (1990) 176.
- [17] A. Belavin, A. M. Polyakov, and A. Zamolodchikov, *Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory*, *Nucl.Phys.* **B241** (1984) 333–380.
- [18] A. Belavin and A. Zamolodchikov, *On Correlation Numbers in 2D Minimal Gravity and Matrix Models*, *J.Phys.* **A42** (2009) 304004, [[arXiv:0811.0450](#)].
- [19] G. W. Moore, N. Seiberg, and M. Staudacher, *From loops to states in 2-D quantum gravity*, *Nucl.Phys.* **B362** (1991) 665–709.
- [20] A. Belavin, B. Dubrovin, B. Mukhametzhanov, *Minimal Liouville Gravity correlation numbers from Douglas string equation*, *JHEP* **1401** (2014) 156. [[arXiv:1310.5659](#)].
- [21] V. Belavin, *Unitary Minimal Liouville Gravity and Frobenius Manifolds*, [[arXiv:1405.4468](#)].
- [22] A. Belavin, V. Belavin, *Frobenius manifolds, Integrable Hierarchies and Minimal Liouville Gravity*, [[arXiv:1406.6661v2](#)].
- [23] V. Belavin, *Correlation Functions in Unitary Minimal Liouville Gravity and Frobenius Manifolds*, [[arXiv:1412.4245v1](#)].
- [24] B. Lee, *G-Frobenius manifolds*, [[arXiv:1501.02118v1](#) [[math.AG](#)]]
- [25] M. Fukuma, H. Kawai and R. Nakayama, *Explicit Solution for $p - q$ Duality in Two-Dimensional Quantum Gravity*, *Commun.Math.Phys.* **148** (1992) 101-116
- [26] P. H. Ginsparg, M. Goulian, M. Plesser, and J. Zinn-Justin, *(p, q) String actions*, *Nucl.Phys.* **B342** (1990) 539–563.